The treatise that follows does not purport to be a complete description of underwater acoustics or of biological sound scattering. It is provided for those who want to know, in a functional manner, how TAPS operates in just enough detail to effectively employ it and analyze the outputs. Space limits preclude an exhaustive description of any facet of acoustics. However, it is our experience that it is not necessary to be able to derive the Naiver-Stokes equations to learn to fly—thus, it should be possible to learn to operate TAPS without a complete knowledge of acoustics, electronics, or signal processing. Within limits, though, we will touch on each of these subjects in an attempt to provide just enough background for the non-physicist to understand what TAPS does and what the results mean.

There are textbooks dedicated to explaining acoustics in more or less detail. Some that we find useful are listed in the references.

The prevalence of acoustic instruments in the ocean arises from the fact that acoustic energy propagates through water much farther, and with lower losses, than any other form of radiant energy (e.g., light). Thus the use of acoustic fathometers for bottom-finding, sonars (and their passive equivalents) for detecting ships and submarines, and echosounders for detecting schools of fish.

Acoustic energy consists of vibrational motion of the water molecules themselves, traveling as a wave through the water. In fluid media (air, water), the direction of the vibrations is in the direction of travel -- these sorts of vibrations are called longitudinal waves. Elastic bodies can support wave motions where the direction of the molecular vibration is at right angles to the direction of propagation -- shear waves. Longitudinal waves in water can sometimes induce these shear waves in elastic bodies immersed in the fluid but these shear waves couple back out of the body into the fluid as longitudinal waves again.

To illustrate some important points about sound and sound propagation, consider a small sphere immersed in seawater. We force the sphere (somehow) to expand and contract in a regular fashion at some frequency, f (in cycles per second or Herz, abbreviated Hz). As the sphere expands, it pushes the fluid away radially. The fluid immediately next to the surface of the sphere has to move radially outward precisely as and when the surface moves. Because the fluid is not perfectly rigid, however, the layer of fluid just a bit off the surface doesn't feel the force until a short time has elapsed, after which it begins to move. The next layer similarly doesn't react until after a short delay, and so on. Thus, the fluid motion takes finite time to react to the motion of the sphere. The force that moves the parcels of fluid is the small increase in static pressure generated by the motion of the adjacent parcel. Thus, there is a radially-expanding pressure wave generated by the expansion of the sphere. This wave moves at a finite speed -- the speed being an intrinsic property of the fluid.

As the rate of expansion slows and the outer surface of the sphere begins to contract, the fluid next to the sphere follows the motion of the surface again (assuming our motion is not too large or rapid). The restoring force that moves the fluid back towards the sphere is the ambient (static) pressure. The fluid a bit farther away takes a little time to feel the new direction of motion, and ... well, you get the picture. A moving wave of negative pressure (relative to the static pressure, that is) is generated in the fluid.

Suppose we drive the surface of the sphere so that the velocity is given by

$$u = U_o \cos(2\pi ft)$$

where f is the frequency of the sinusoidal motion of the surface, t is time, and U_o is the magnitude of the surface velocity. It can be shown that the pressure, P, corresponding to this velocity is

$$P = \frac{-\rho cka^2 U_o}{r} \sin(2\pi ft - kr)$$

where a is the radius of the sphere, ρ is the density and c is the sound speed of the fluid, r is the range at which the pressure P is measured, and k is the wavenumber, $k=2\pi/\lambda$ where λ is the wavelength, $\lambda=c/f$.

Some things about this relation are worth noting: First, the magnitude of the pressure is dependent upon the properties of the fluid. Vibrating the same sphere in air would produce a much smaller pressure than it will in water. Next, the pressure described by this equation is a travelling wave, as shown by the argument of the sin function. A given value of P will occur for a constant value of 2πft-kr. That is, whenever 2πft-kr is equal to some fixed value, we will find the same value of P. The only way this can happen is if r increases with time. The pressure value travels radially outward at the rate k/2πf. Looking at the definition above, this is simply equal to c, the speed of sound. Third, this wave is a spherical wave expanding radially outward and the pressure decreases as range increases.

The intensity of a travelling wave is defined as the rate of transfer of energy per unit area per unit time. It is the flux of power through unit area. The intensity of the pressure wave from our little sphere can be found from

$$I = \frac{P^2}{2\rho c}.$$

Comparing these last two equations, you should be able to see that the intensity of the sound wave is inversely proportional to the square of the distance away from the source. This can be shown another way as well. Suppose we supply W watts of power to our source. This power is radiated in all directions equally. At some range, r, then, we should have a flux of W watts through an area, $A = 4\pi r^2$. Thus,

$$I = \frac{W}{4\pi r^2}$$

so it is clear that the intensity must decrease as the inverse of distance-squared. This decrease in intensity is called spherical spreading and is purely a geometric phenomenon.

Except for a few rather pathological cases, all sound waves in the ocean can be treated as spherical waves—at least at some distance from the source.

We can use the equations developed so far to put some numbers to these quantities. Suppose we supply 4π watts (about 12.6 W) to our sphere. (This is actually quite a lot of power, on the order of a very loud boom-box.) We can solve to find[1]

$$p^2 = \rho cI.$$

Now the density of sea water is about 1025 kg/m^3 and the speed of sound is about 1500 m/sec. If we calculate the pressure at a range of 1 m, where the intensity is 1 W/m^2, the resulting pressure is about 1240 Pa (Pascals, or Newtons/meter^2). For reference, atmospheric pressure is about 101,000 Pa.

In underwater acoustics, we tend to use logarithmic quantities whenever possible. In particular, we use the decibel

[1] We have changed terminology here. The peak pressure amplitude of the wave is P, as before; the symbol p is used to denote the root-mean-square (RMS) value of the pressure wave as this measure corresponds to power and intensity units. For example, the published voltage for common household AC in the US is 117VAC. If you were to look at the waveform on an oscilloscope, you would find the peak voltage is about 165V (plus and minus). The published value is the RMS or effective value. The product of RMS volts and RMS amps is power in watts.

to describe acoustic intensities and quantities derived from intensities. The decibel equivalent of an intensity, I, is given by

$$L = 10\log\left(\frac{I}{I_{ref}}\right)$$

where the Naperian log (base 10) is used. Note the presence of a reference intensity, I_{ref}. Logarithms can only be taken on non-dimensional quantities, hence the intensity must be divided by another intensity. Decibel intensities are often referred to as 'levels'.

The reference level for underwater acoustics is a pressure wave with a RMS amplitude of 1μPa. A reference pressure can be used instead of a reference intensity because the terms ρc will cancel. So we can write an equivalent expression for the acoustic level

$$L = 10\log\left(\frac{\rho c p^2}{\rho c p_{ref}^2}\right)$$

or

$$L = 20\log\left(\frac{p}{p_{ref}}\right).$$

If we insert the pressure we calculated for our 12.6 W spherical source, we obtain

L=+181.9 dB//μPa

where the units following the value mean dB re (or referenced to) 1 micro-Pascal.

The calculations we have done to this point provide us with a value for the acoustic pressure corresponding to an intensity of 1 watt/m^2. More important, however, we have derived an important descriptor of acoustic transducers: Source Level. The Source Level (SL) of a transducer is defined as the acoustic level, expressed in dB//μPa, measured at (or referred to) a distance of 1 meter from the transducer. It is usually a measured quantity (see the calibration sheet for your TAPS in the Appendix I).

With the SL for our source, together with the rule for spherical spreading, we can now calculate the acoustic pressure or intensity at any point in the field around our source.

Now let's put a small object in the fluid at a range, r, from our source. Acoustic waves travel from the source to this object (let's call it a target), strike the target, and proceed onwards towards infinity, getting weaker and weaker due to spherical spreading. Some of the acoustic energy is scattered by the target, however, and travels back to the source. If we pulse the source -- turn it on for a short time, τ, and then back off again -- the packet of waves could travel out to the target, some fraction of the energy be reflected, the packet travel back to the source, and we could measure the elapsed time to estimate the range to the target. This is the principle of a fathometer or echosounder, measuring time delays to estimate depth to the bottom or to a fish.

It is simple to calculate that the acoustic intensity and pressure at the target will be

$$I(r) = \frac{I_o}{r^2}$$

and

$$p(r) = \frac{p_o}{r}$$

where I_o is the intensity and p_o the pressure at 1m distance from the source. Of the acoustic energy striking the target, some fraction will be scattered back towards the source. We can define a reflectivity, R, in the following way,

$$\mathbf{R} = \frac{p_{scatt}}{p_{inc}}$$

where p_{inc} is the pressure incident on the target (we just calculated this) and p_{scat} is the pressure scattered from the target measured at (or referred to) a distance of 1 meter from the target in the direction of the source.

Note that the reflectivity is defined as a ratio of pressures (and we already know how to convert these to intensities if we wish). Thus, it should come as little surprise to find that acousticians have taken the log of this and given it a name: Target Strength, viz.

$$TS = 20\log\left(\frac{p_{scatt}}{p_{inc}}\right)$$

or

$$TS = 20\log(\mathbf{R}).$$

We can now write an expression for the scattered pressure from our target, measured back at the source location (remembering that the signals reflected back to us undergo spherical spreading also):

$$p_{meas} = \frac{p_o R}{r^2}$$

or, in terms of logarithms,

$$EL = SL - 40\log(r) + TS$$

where EL is the Echo Level. This equation is one form of the Sonar Equations first commonly employed by Robert Urick.

Let’s put some numbers to this equation to see what sorts of pressures we are talking about. Suppose our ‘target’ is a copepod. A typical TS for a small copepod might be -110 dB. This means that the ratio of scattered pressure to incident pressure is about 3.2×10^{-6}. Mighty small. If we used the source described above and put the copepod one meter away, then the Echo Level would be about +71.9 dB//µPa. Converting this to pressure, the echo signal would be about 3918 µPa. For reference, this is about 1/10th the level of thermal noise in the water at a frequency of 1 MHz[2]. So this source would not be useful in detecting single copepods.

Real echosounders (and TAPS) don't use spherical sources, though. Most often, the transmitting transducer is a planar shape – circular disk or a rectangle -- that preferentially directs more energy in a particular direction and, as you might deduce, less in other directions. Let's add directivity to our simple example as shown in Fig. 1. The curvy pattern drawn around the source point indicates the amount of acoustic intensity transmitted in each direction. Note that the levels are in dB. We have drawn this figure so that the angle of maximum intensity points at the target. Thus, if we characterize the source by the intensity transmitted on the Major Response Axis (MRA, the angle at which the intensity is a maximum), then the equations developed above still apply.

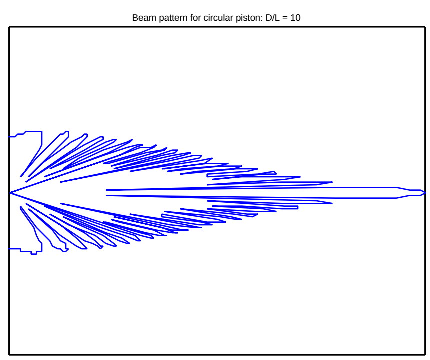

Figure 1. Beam pattern of a circular disk transducer with diameter = 10 λ.

With real transducers, one of the effects of directivity is an increase in SL (on the MRA) for a fixed input power. This effect is characterized by a sort of ‘gain’ function called the Directivity Index, or DI. One can calculate the Directivity Index from a measured beam pattern or, in certain cases, from knowledge of the geometry of the transducer. The value of DI will be

[2] And in the bandwidth of TAPS. See Urick, Chapter 7 if you want to learn more about ambient noise.

proportional to the ratio of on-axis intensity to the intensity of an omni-directional source producing the same power output. Note that this gain function applies to reception as well.

Interestingly, one can quite accurately compute the Source Level for a transducer from a knowledge of the acoustic power, P_a (or the input electrical power, P_e and the conversion efficiency, η) and the Directivity Index. The equations are

$$SL = 170.7 + 10\log(P_a) + DI$$

and

$$SL = 170.7 + 10\log(\eta P_e) + DI$$

where the constant, 170.7, accounts for the properties of the fluid. In this case, the fluid is assumed to be sea water.

Suppose we smashed up our little sphere and molded it instead into a circular disk. Let us suppose that the face of the disk is, say, 10 wavelengths in diameter. (It happens that the directivity or beam patterns of apertures—acoustic, optic, or electromagnetic—are functions of their size computed in terms of the wavelength of the radiated signals[3].) Now we can compute that the expected DI of this disk will be approximately 30 dB. If we put the same electrical power into the disk, our Source Level will increase from +181.9 dB//μPa to +211.9 dB//μPa. The pressure (on axis) at 1m will increase from 1240 Pa to 39,200 Pa. Of course, you don't get something for nothing. The pressure at an angle[4] of, say 4.5° will be practically undetectable and at 7° will have risen again to only about 4 times the original level from our sphere. The pressure will fall and rise again with increasing angle until it is essentially below ambient noise.

We can call the 'gain' function b(θ), where θ is the angle from the MRA. The pressure at θ will be $p_o b(\theta)/r$. Note that the transducer is reciprocal, in the sense that the response of the transducer to a pressure wave arriving at the angle θ will have the same 'gain' function compared to pressure waves arriving on the MRA. The directivity function, b(θ), has a maximum at θ= 0° normally, where b(θ) = 1. The directivity would be less than 1 everywhere else.

The Echo Level from our copepod as seen by this transducer will now be 30 dB higher, or +101.9 dB//μPa. The backscattered pressure (at 1 m) would be about 123,900 μPa, well above ambient noise. So directivity helps detectability. But put the copepod somewhere else in the beam and the Echo Level will probably decrease again to undectable levels.

But, when was the last time you saw one copepod all by itself? The little varmints hang out in gangs of thousands or more. So the chances of our seeing only one copepod in our beam is pretty remote. We are more likely to have 10's to 1000's in the beam at one time. How do we deal with the scattering from these? Well, with more math.

Let's digress a bit, first, to look at how pressure waves interact. Suppose we have two copepods—let's call them 1 and 2—at slightly different angles and slightly different ranges. Each will scatter sound back towards the source. The signals scattered by each could be written as

$$p_1 = \frac{p_o b^2(\theta_1) R_1}{r_1^2} \sin(2\pi f t - k r_1)$$

and

$$p_2 = \frac{p_o b^2(\theta_2) R_2}{r_2^2} \sin(2\pi f t - k r_2)$$

where r_1, r_2 are the ranges to the two scatterers; R_1,R_2 are their reflectivities; and $b^2(\theta_1)$ and $b^2(\theta_2)$ are the (combined) response of the transducer at the respective

[3] See, for example, Urick Chapter 3 or Clay and Medwin Chapter 5.2 or Medwin and Clay Chapter 4.2-4.4.

[4] Because of the circular symmetry of this shape, the only dependence is on the angle from the MRA. Other transducer shapes might depend on <u>two</u> angles, azimuth and elevation, say.

angles to the copepods. We have used sine functions to express the time-dependence of the signals. The two signals will add together at the receiving transducer, producing a single signal that cannot (normally) be distinguished from the scattering from a single target. Except, these signals are sinusoids and they add with a phase-shift. That is, the peaks of one signal may or may not coincide with the peaks of the other signal. If they do, the sum signal can be as much as twice the amplitude of either signal alone. If the positive peaks of one coincide with the negative peaks of the other, the signal may disappear altogether except at the very ends of the signal. It all depends on the exact difference in ranges to the two copepods.

Extending this idea to N copepods, it is obvious that we will have to represent the scattering from a bunch of scatterers as a sum over the individual positions and reflectivities. We can express the total scattered pressure as

$$p_t = \sum_{k=1}^{N} \frac{p_o R_k}{r_k} \sin(2\pi f t - k r_k) .$$

where N is the (unknown) number of copepods with (unknown) ranges r_k and (unknown) reflectivities R_k.

We can simplify this equation a little. It is generally assumed that the range is great enough that the small variations of the r_k do not change the result significantly in the amplitude term (they are cruicial to the phase term inside the sine function, of course) so we can pull the range dependence out of the summation, viz.

$$p_t = \frac{p_o}{r} \sum_{k=1}^{N} R_k \sin(2\pi f t - k r_k)$$

All of the remaining quantities inside the summation as well as the number of scatterers, N, would be expected to vary randomly from ping to ping. **This is the defining characteristic of volume scattering—the echo from a given range has a random amplitude and phase on each ping**.

Now there is a result from signal physics that will help considerably here. Suppose we add together the outputs of some number, N, of independent oscillators all running at the same frequency. If we set the amplitudes of each oscillator to the same value (1 volt, say) and adjust the phases so that all the oscillators are in phase with each other, the sum voltage will be N volts. If we add the outputs of N independent oscillators, all of which are at random phases with each other, the sum voltage will be around $\sqrt{N}$ volts. I say 'around' because the exact voltage will depend on the exact sets of phase shifts of the oscillators. But—and here we get into signal physics—if we repeated the experiment many times, each time setting the phases of the oscillators to new random values, the average value of the sum voltage will eventually settle out at exactly $\sqrt{N}$ volts. That is, the expected value of the sum of N independent, random signals of the same frequency and amplitude is $\sqrt{N}$ times the amplitude of one oscillator.

If we shift our attention to the echo intensity, which is proportional to the square of the pressure, we find that the expected value of the intensity of the sum signal is exactly N times the intensity of one oscillator's output. We write this as

$$\langle I_{sum} \rangle = N I_{one}$$

where the <...> brackets mean 'expected value of' in the sense of the average of a set of independent trials (ensemble average).

Looking back at the situation where we have a random number of copepods of random target strengths located at random positions, we can see that all we can hope to model is the expected value of the echo level based upon expected values of the various parameters. Anticipating a result from the

Appendix to this appendix, we can state that the expected value of the echo intensity will be proportional to

$$\langle I \rangle \quad \frac{I_o}{r^4} \langle N \rangle \langle R^2 \rangle \langle B^2 \rangle$$

where <N> is the mean number of copepods in the scattering volume, $<R^2>$ is the mean-squared reflectivity of the copepods (or average target strength in intensity units), and $<B^2>$ is a beam pattern factor that accounts for the random locations of the copepods in the beam. Note that this relationship holds only when N is a large (in the statistical sense) number.

The good news is, the mean backscattered intensity from a region containing critters is proportional to the number of critters in that volume. The difficulty lies in those pesky other factors.

We can make some headway if we assume that the critters are uniformly-randomly distributed in space—or at least in the region causing the scattering. We can then re-write <N> as (see page 19) ηV, where η is the average density of scatterers (per cubic meter) and V is the ensonified volume. This allows us to revise the equation above as

$$\langle I \rangle \quad \frac{I_o}{r^4} \eta \langle R^2 \rangle V \langle B^2 \rangle .$$

The factor $V<B^2>$ can be thought of as the effective ensonified volume. V is the volume of the spherical shell delimited by the leading and trailing edges of the transmitted pulse. The range limits of the shell are r to r + cτ /2, where τ is the pulse length and c the sound speed.

Under the assumption of uniform random distributions, we can express the effective ensonified volume as an integral in the following form:

$$V < B^2 >= \int_V BB' \, dV$$

where B is the beam pattern function and V is the volume. We can express the differential volume in spherical coordinates as

$$dV = r^2 \frac{c\tau}{2} d\Omega$$

where $d\Omega$ is the spherical solid angle. Thus,

$$V < B^2 >= r^2 \frac{c\tau}{2} \int_0^{4\pi} BB' \, d\Omega \, .$$

It is necessary to specify the transducer beam pattern to go any farther.

TAPS uses circular piston transducers. The integral for this geometry can be evaluated explicitly. The result is

$$\int_0^{4\pi} BB' \, d\Omega \quad \frac{4.853}{kD}$$

where D is the diameter of the transducer and

$$k = \frac{2\pi}{\lambda}$$

$$k = \frac{2\pi f}{c}$$

is the wavenumber (and here λ is the wavelength). So our estimate of the effective ensonified volume is

$$V_e = r^2 \frac{c\tau}{2} \frac{4.853}{kD}$$

and our expression for the backscattered intensity becomes

$$\langle I \rangle \quad \frac{I_o}{r^2} \eta \langle R^2 \rangle \frac{c\tau}{2} \frac{4.853}{kD}$$

where the r^2 of the volume term canceled some of the $1/r^4$ spreading loss.

The factor $\eta<R^2>$ cannot be calculated—it is usually the factor we want to measure with our system. It is so important that we give it a name of it's own. The mean-squared reflectivity of a unit volume of water containing scatterers is called the volume backscattering strength, Sv. Thus we can write

$$Sv = 10\log(\langle R^2\rangle) + 10\log(\langle\eta\rangle)$$

or the Volume Backscattering Strength is equal to the mean target strength of the critters plus ten times the logarithm of the average number of critters in 1 cubic meter.

If we define s_v as the linear equivalent of Sv, then

$$s_v = \eta \langle R^2\rangle.$$

The volume backscattering coefficient is equal to the mean density of scatterers times the mean-squared reflectivity of the scatterers.

At this point, it is useful to convert the backscattered intensity into what we actually can measure – the voltage out of our receiver. Our transducer has a conversion efficiency in transducing the pressure waves to voltage; call this factor M. Then the output voltage for an input pressure wave of amplitude P is

$$V = MP$$

Our receiver amplifies the voltages out of the transducer by about a factor of 4000 (give or take). We will assume here that this gain is incorporated into M.

Thus we can re-write our equation (for almost the last time) as

$$\langle V^2\rangle \quad M^2 \frac{I_o}{r^2}\eta\langle R^2\rangle \frac{c\tau}{2}\frac{4.853}{kD}.$$

Finally, we take logarithms of this equation to cast it in the Sonar Equation form,

$$\bar{V} = SL+RS-20\log(r)+Sv+K$$

where

$$\bar{V} = 10\log(\langle I\rangle)$$
$$= 10\log(\langle V^2\rangle)$$

is the mean-squared voltage ouput of the transducer over many independent pings, SL is the Source Level of the transducer,

$$RS = 20\log(M)$$

is the Receiving Sensitivity (plus gain) of the transducer, r is the range to the region causing scattering, and Sv is the Volume Backscattering Strength. The factor K is given by

$$K = 10\log(\frac{c\tau}{2}) + 10\log(\frac{4.853}{kD})$$

or

$$K = 10\log(\frac{c\tau}{2}) + 7.7 - DI$$

where DI is the directivity index* for our circular piston transducers. K takes care of adjusting for the fact that the actual scattering volume is not 1 m^3 and *this factor embodies the assumptions we made above*: (1) *there are a large number of scatterers* and (2) *the scatterers are uniformly-randomly distributed*.

Solving this equation for Sv, we have

$$Sv = \bar{V} - (SL + RS - 20\log(r) + K)$$

or

$$Sv = \bar{V} - K_1$$

where K_1 is a (range-dependent) system constant. This is the equation buried inside TAPS-6. $\bar{V}$ is what TAPS measures, Sv is what TAPS outputs.

It's time to step back and focus on something that hasn't been emphasized enough in the foregoing derivation. **Volume scattering is a random**

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* See Urick, 1967 and see the calibrations sheet for your TAPS for values

process. One ping provides absolutely NO information. The measurement that we can relate to the abundance of scatterers is the mean value of the scattered intensity. Like all random variates, this implies that our measurements will approach closer and closer to the true mean value as we average more and more samples.

One way to obtain more samples is to sample echo intensities at the same range over multiple pings. Pinging madly, we could eventually arrive at an estimate that was extremely close to the mean value we desire. Until, that is, things changed.

Implicit in the notion of a mean value is the concept of stationarity. In order for a mean value to exist, the underlying statistics of the process must remain the same. For our Volume Backscatter Strength to have any meaning, all of the measurements must be taken from essentially the same mix and average abundance of scatterers.

If you image a typical vertical profile of zooplankton – especially in light of recent evidence of thin, intense layers of phytoplankton and zooplankton on the order of 10 cm – it will become clear that this is a serious problem in acoustic assessment. Sharp gradients in abundance, such as near a thermocline, mean a non-stationary population. How do we manage to get decent echo statistics in such a situation?

Well, the need is for **statistically independent** samples of the back-scattering. Samples from successive pings are probably independent because of the passage of time between pings, so that the exact configuration of scatterers (relative phases) has changed enough to make the correlation between configurations small. There is another way to get independent samples, however.

At any instant of time, t, the echo intensity is composed of scattered signals from scatterers located in a spherical shell of thickness cT/2. That is, the leading edge of the transmitted ping has travelled ct/2 meters out and ct/2 meters back to the transducer. The trailing edge of our transmit pulse (of duration T) has travelled c(t-T)/2 meters out and c(t-T)/2 meters back to the transducer. So the echoes that are making up the echo intensity at this instant arise from a shell whose farthest edge is ct/2 away and whose nearest edge is c(t-T)/2 away; thus the thickness is cT/2.

At time t+T/2, half a transmit pulse length later, the near edge of the shell of scatterers is at ct/2 and the far edge is at c(t+T)/2 away. And this shell contains NO scatterers from the previous shell. Thus, echo samples at intervals of T/2 are independent samples. Aha!

In CAST MODE, TAPS takes advantage of the independence of sequential samples by sampling several times on each ping and averaging the intensities of these samples. Averaging again over N pings increases the number of independent samples even more. Then, in data processing, binning the data into range bins averages several data sets to obtain even more samples so that the mean value tends closer to the ‘true’ value. Thus, CAST MODE data can get by with relatively few pings per average—on the order of 4-12, say.

In SOUNDER MODE, TAPS averages echo intensities at discrete range intervals; each sample is the average intensity measured at fixed range over the N ping cycles. Since the only averaging is done over pings, we need a fairly large number of pings—16-32 or even more—to obtain reasonably accurate mean values.

So what’s to stop us from taking several hundred pings to get even better averages? Well, nature. Or more properly, stationarity. In CAST mode, it is extremely unlikely that the vertical structure of scattering is unchanging for meters at a time. In fact, recent work has shown orders of magnitude changes in biomass over 10’s of cm in depth in some places. Detection and enumeration of such thin layers requires very slow descents and rapid sampling times (or relatively few pings per average).

Horizontal coherence distances for zooplankton populations are not well characterized but our experience with SOUNDER data suggests that abundances can change significantly over time intervals of minutes—again suggesting rapid sampling and relatively few pings per average are necessary to be able to observe the variations that exist in nature.

So there is a trade-off to be made between accuracy and resolution. This is a choice you have to make. Keep this fact in mind, however. It is always possible after the fact to bin data to increase the statistical accuracy. It is never possible after the fact to pick the data apart to increase the spatio-temporal resolution.

INVERSION

Let's return to the equation for s_v. This time, let us suppose that there are N_1 scatterers of reflectivity R_1, N_2 scatterers of reflectivity R_2, and so on. Let us write I' for the average relative backscattered intensity (corrected for the incident intensity and range losses). Then we can write

$$I' = N_1 R_1^2 + N_2 R_2^2 + \ldots + N_n R_n^2$$

where we assume there are **n** different classes of scatterers. Note that even if we knew the mean-squared reflectivities for all of the scatterer classes, we could not find the numbers of critters in each class. At best, we can find the total number of scatterers if we know the overall mean-squared reflectivity for all the scatterers.

But, suppose we make separate measurements of the backscattered intensities at, say, **m** different frequencies. Then we could write the set of equations,

$$I_1 = N_1 R_{11}^2 + N_2 R_{21}^2 + \ldots + N_n R_{n1}^2$$

$$I_2 = N_1 R_{12}^2 + N_2 R_{22}^2 + \ldots + N_n R_{n2}^2$$

...

$$I_m = N_1 R_{1m}^2 + N_2 R_{2m}^2 + \ldots + N_n R_{nm}^2$$

Now we may be on to something. This set of equations relates a set of measured values (the relative backscattered intensities) to a set of unknown quantities (the numbers of scatterers in each class) through a set of possibly-knowable parameters (the mean-square reflectivities or target strengths). If we can somehow come up with the R's, it may be possible to solve for the N's.

Actually implementing a solution entails a number of problems. In no particular order, these problems include: (1) coming up with a scattering model so that we can calculate the R values; (2) selecting the set of frequencies—number and values—that will best estimate the abundances of some range of critter classes; (3) developing a computational solution method that is sufficiently accurate and rapid; (4) understanding the sources and effects of errors.

METHODS

The equations derived above form a linear set of **m** equations in **n** unknowns. Given plausible estimates for the scattering strengths of the various sizes, R_{ij}^2, it ought to be possible to estimate the vector of sizes, N by some form of inversion.

Solution of sets of linear equations is a branch of mathematics that provoked a lot of interest in the 1970's (in search of solutions to similar problems arising out of geophysics). Several methods for dealing with this set of equations were developed, some with very interesting names (Most Squares, Edgehog, etc.).

In solving any set of equations, one is first interested in whether or not a solution exists at all, then with the uniqueness of the solution (or solutions), last with the quality of the answer.

Our set of equations resemble a discrete version of the Fredholm Equation of the first kind. This equation has the unhappy distinction of being what is called an ill-posed problem. Basically, this means that simple solutions to these equations can include highly magnified versions of the inevitable input errors. Our goal, then, is to find a way to generate solutions that do not magnify errors any more than is absolutely necessary.

The familiar technique of Least-Squares is one method for obtaining a solution while minimizing errors. By a complicated path, one obtains a single solution that minimizes what is called the Euclidean length of the unknown vector, ||N||. If, that is, there are as many as or more equations than unknowns (m≥n).

Since we are limited to six frequencies in the TAPS-6, this would limit us to solving for at most five size classes (plus total abundance).

It is possible to solve our set of equations when there are more unknowns than equations (m<n). The problem, however, is that there is more than one solution. There are infinitely many solutions! Oh, my. This is not good.

But, of course, there is some hope. The original problem can be written in matrix form as

$$I = R\,N$$

where I is the vector of measured backscattered intensities, R is the matrix of scattering model squared-reflectivities, and N is the (unknown) vector of abundances vrs size (size-abundances). For m<n this problem has too many solutions. But what if we solve another, hopefully similar problem instead:

$$\begin{bmatrix} I \\ I_o \end{bmatrix} = \begin{bmatrix} R \\ R_o \end{bmatrix} \begin{bmatrix} N \\ N_o \end{bmatrix}$$

where we have added rows to the vectors and scattering matrix to make n=m. We can take

$$R_o = \lambda \boldsymbol{I}$$

where $\boldsymbol{I}$ is the identity matrix and here we are using λ as a constant. We generally take Io to be zeroes which expresses a preference for the minimum-length solution (fewest number of scatterers to fit the measured data). Thus the problem becomes

$$\begin{bmatrix} I \\ 0 \end{bmatrix} = \begin{bmatrix} R \\ \lambda \boldsymbol{I} \end{bmatrix} [N]$$

This technique, known as the Levenberg-Marquardt method, lets you tailor a solution via the parameter, λ. And since n=m, we can get an ‘unique’ solution—one for every value of λ we try. It turns out that solutions using large values of λ produce solutions with small values for N but fairly large discrepancies between the input data and intensities calculated from the solution vector, N, and the model. Small values of λ produce solutions with larger values for N and smaller discrepancies.

Conceptually, one desires to jointly minimize the size of the solution vector (a sort of Occum’s Razor principle) and the estimated error (RNORM in our Matlab version of the inverse). In practice, one normally finds a reasonable value that gives pleasing results and uses it until circumstances force another look.

Let’s take a more detailed look at R, the scattering model matrix. We hinted that the R terms in the equation

$$I' = N_1 R_1^2 + N_2 R_2^2 + \ldots + N_n R_n^2$$

were the same model and differed only in the size term. In fact, there is nothing to limit the scope of the models. We can build a model matrix that consists of several models combined. We can re-write this equation to make this explicit:

$$\Gamma = N_1 R_1^2 + N_2 R_2^2 + \ldots + N_{n1} R_{n1}^2 + N_{n1+1} S_{n1+1}^2 + N_{n1+2} S_{n1+2}^2 + \ldots + N_{n1+n2} S_{n1+n2}^2 + N_{n1+n2+1} T_{n1+n2+1}^2 + N_{n1+n2+2} T_{n1+n2+2}^2 + \ldots + N_{n1+n2+n3} T_{n1+n2+n3}^2$$

where R is one sort of model, S models another organism, and T is yet another model. The first n_1 terms of the solution vector, N, would be the abundances of organisms modelled by R, the next n_2 terms would be the abundances of organisms modelled by S, and the last n_3 terms the abundances of the organsims modelled by T.

Simple solutions to these equations using least-squares can produce some surprising results. It is quite possible to obtain negative numbers of scatterers at some sizes. While mathematically satisfactory, this sort of result is clearly unrealistic.

We can add a constraint to the problem without affecting the accuracy of the solutions,

$$N_i \geq 0$$

for all i. One algorithm for solving the set of equations with this constraint is called Non-Negative Least Squares (NNLS; Lawson and Hanson, 1974). This algorithm has been incorporated into Matlab and is the one we use in our inversion programs.

SCATTERING MODELS

There have been a couple of methods used to develop scattering models for marine organisms. Physics-based model development generally starts with some simplified assumptions about the physical composition of the organism, represents the organism shape in some mathematically simplified way, and then solves this abstract model for it's acoustical scattering properties. Validation of such a model usually involves comparing the model predictions to laboratory measurements of target strength. Examples of physics-based models range from the purely geometric fluid sphere model (a version of which we use for copepods and similarly-shaped zooplankters) through the adaptive-geometry of Tim Stanton's truncated, rough fluid cylinder model for euphausiids.

Empirical model development, on the other hand, relies almost completely on actual measurements and uses modeling methods largely to organize and parameterize the empirical data. The best example of this is Love's (1977) models for Target Strengths of fish.

Anderson (1950) published a solution for scattering from a fluid sphere and suggested that this model might find use in predicting target strengths for marine zooplankters. We will use this model as an example of physics-driven model development (and also because we use a variant of this quite heavily).

Anderson's model solved for the scattered wave from the fluid sphere in terms of 'modes' of vibration of the sphere. That is, he solved the differential equations for the scattering problem in terms of a sum of oscillatory terms that can be shown to have physical analogs with simple radiators.

You are already familiar with one mode from our simple source. As the sphere expands and contracts, sound is generated in the surrounding fluid. This mode, the "zeroth" order or Monopole mode, can also be excited by sound impinging on the sphere. As you might imagine, the amount that the sphere will contract and expand, as the pressure wave passing by it goes positive and negative, depends upon the relative compressibility of the sphere compared to water.

But unless the sphere follows the motion of the surrounding water precisely, the difference between the actual contraction and expansion of the sphere and the contraction and expansion it would have experienced were it composed of water, can be thought of as a source of sound waves. Thus, we say that the sphere *reflects* some of the incident sound.

Softer spheres made, say, of air, will contract and expand much more than would spheres made, say, of steel. Thus, the sound radiated by the Monopole mode of scattering depends on the compressibility of the sphere compared to water.

The next mode of oscillation (and scattering) is due to the forces applied to the sphere by the pressure wave causing accelerations in the direction of the sound wave's travel. The water particles move as the sound wave passes by, sloshing gently to and fro as the local pressure excess goes positive and negative. The sphere also partakes of this motion but may move more or less as far and more or less as fast as the water particles depending upon the relative density of the sphere compared to the water. A dense sphere will move more slowly than and lag behind the water particles, a light sphere would move more rapidly and race ahead of the water particles. In fact, a bubble of air can move 3 times as fast as the water under ensonification.

Again, the difference between the actual motion of the sphere and the motion it would have experienced were it made of water can be thought of as a source of sound waves. This scattering mode is known as the Dipole mode.

Higher modes of scattering exist. These modes involve oscillations of the surface of the sphere such that an integral number of oscillations will fit on the surface (while the sphere is expanding and contracting and moving back and forth!). These modes depend critically upon the spherical geometry of the scatterer and the

Figure 2. Target Strength of a 1mm copepod versus frequency. The upper curve is the truncated Anderson fluid sphere model that can be used to model scatteres such as copepods; the lower curve is the full modal expansion of this model applicable to perfect spheres.

frequency of the incident sound waves. If a scatterer is not spherical, however, it is unlikely that these modes will exist. If the scatterer is sort of spherical (not significantly elongate, for example), then is is probable that the low-order modes (0,1) would exist but that the higher-order modes would not. This is the thinking that led us to try a truncated fluid sphere model as an analog for small zooplankters like copepods.

The amplitude of the scattering from a fluid sphere (truncated or full) depends upon several factors: the size of the sphere (radius, a) relative to the wavelength (λ) of the incident sound, the relative density of the sphere to that of the surrounding water ($g = \rho_s/\rho_w$), and the relative compressibility of the sphere to that of the surrounding water ($h = \kappa_s/\kappa_w$).

The compressibility of a fluid, κ, can be expressed in a different form, viz

$$\kappa = \sqrt{\rho c^2}$$

where ρ is the density and c is the sound speed. Substituting this expression into that for h, we get the pair of relations

$$g = \rho_s/\rho_w$$
$$h = c_s/c_w.$$

So the reflectivity of the sphere is a function of the relative density and sound speed of the sphere to that of the surrounding water.

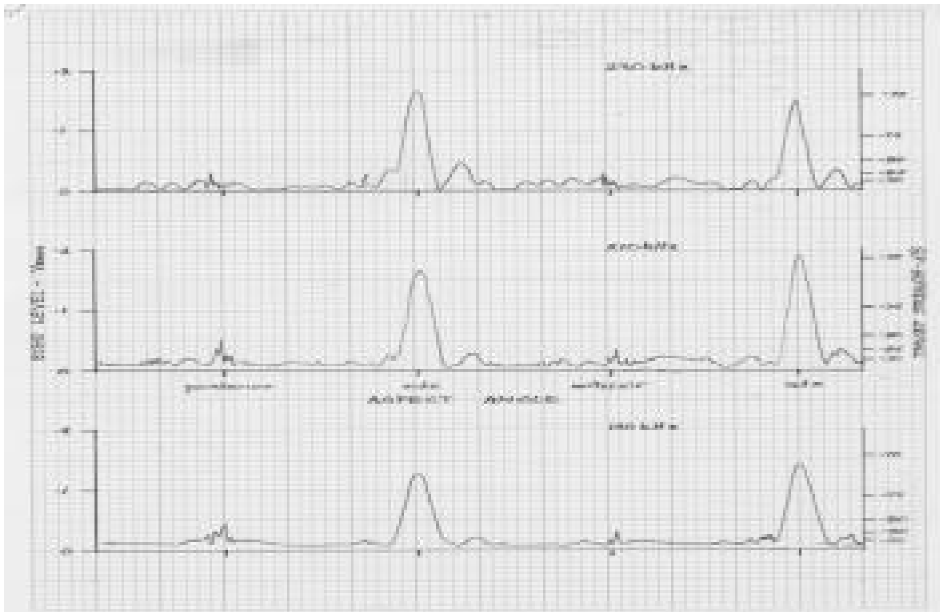
But what if the sphere is not fluid-like? We define a fluid by it's inability to support shear waves – waves that have particle motion across the direction of wave motion. Materials that do support shear waves are called elastic. Most zooplankton appear to be more fluid-like than elastic but there are certainly counter-examples. Think, for example, of pteropods. Models for elastic scatterers exist, both geometric and for marine organisms. These require another physical property (in addition to size, g, and h) to represent the conversion of longitudinal waves – the ones we have been considering up to this point – into shear waves and vice versa.

Or what if the scatterer is not a sphere? Elongate organisms such as euphausiids, mysids, shrimp, chaetognaths, etc. are common in the ocean. Models for these organisms do exist, however they are more empirical in nature due to the non-geometric shapes of these organisms.

A clever method for approximating the backscattering from odd-shaped fluid-like organisms was developed by Tim Stanton (now at WHOI). Based upon the Born Approximation, he came up with a way to estimate the scattered field from irregular, elongate fluid scatterers like euphausiids and shrimp. Guided by measurements, he was able to construct models for backscattering of realistic shrimp. He has also constructed models for elastic scatterers such as pteropods. References to his papers are included below.

One feature of interest in backscattering from elongate scatterers is their directionality. That is, the Target Strength depends upon the angle at which the scatterer is viewed. The figure below shows echo amplitudes for a preserved euphausiid measured in a test tank. The specimen was attached to a fine nylon line that was attached to a rotator. Echoes were sampled as the specimen rotated and the results plotted on chart paper (I took these data in 1980!).

The maximum values for TS occur at side aspect when the axis of the animal is at right angles to the transducer. Everywhere else, the TS is 20 dB or more lower. Now this particular specimen was selected for its straightness; live euphausiids are rarely this straight. But the point should be clear—models for elongate scatterers are going to have to include some parameters



Measured echo amplitudes for a 19.8mm preserved *Euphausia pacifica* at three frequencies as a function of aspect angle. Estimates of Target Strength in dB are given on the right abcissa.

that account for the directional properties of these scatterers.

One thing to keep in mind: TAPS in CAST mode measures the scattering from about a 2 liter effective volume. In a volume of seawater of this size, one might expect to find 10’s to 100’s or more of copepods. In very dense swarms of euphausiids, one might see one euphausiid every 4-10 samples; normal distributions might produce one every 25-100 samples. Larval fish might appear in one out of 100-1000 samples, and so forth. The small sample volume of TAPS tends to exclude the rare larger scatterers from the measurements – BY DESIGN.

This is not the case in SOUNDER mode, however. Sample volumes can be a cubic meter or more at long range and scattering can be dominated by fish, mysids, or similar organisms. In these cases, you must choose your models with care and account for any behavior-driven factors such as average aspect angle.

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VOLUME SCATTERING DERIVATION

(an appendix to Appendix 2)

Canonical theories of reverberant echo formation have been developed by several authors (Cron and Schumacher, 1964; Faure 1964; Ol'shevskii 1964; Middleton 1972). These theories are similar, being based upon similar assumptions, but differ in the degree of complexity of the results. The goal of those studies was to estimate statistical and/or spectral properties of volume reverberation; detailed expansions of the fundamental equations were not presented. In this note, the physical scattering equations are derived and extended to express expected values for higher moments of the reverberation echoes. These expressions are compared with the theoretical expectations for the important cases of few and many scatterers.

Consider a single, isolated scatterer of reflectivity R, located in an infinite homogeneous medium at the point (r,θ,ø) in some set of spherical coordinates. We insonify this scatterer with a tone burst of duration τ, amplitude P_o B'(θ,ø)/r, and frequency f. The term B'(θ,ø) describes the directivity of the transducer, which is located at the origin. Neglecting absorption losses, the echo voltage at the output of the receiving transducer, with sensitivity M, at time t is

$$V(t) = M \bullet P_o \bullet R \bullet B(\theta,\phi) \bullet \exp -j(\omega t - kr)/r^2$$

for times t such that cτ/2 < r < c(t+τ)/2. The beampattern functions for transmit and receive are combined into the factor B(θ,ø). Also, ω=2πf, and k=2π/λ, where λ is the wavelength. Standard notation for complex variables is assumed. In particular, if we write

$$V(t) = A \exp -j(\omega t + \Phi)$$

then the actual voltage is a sine wave of RMS amplitude A and phase Φ. The value of V(t) at any instant lies between $-\sqrt{2}$ A and $+\sqrt{2}$ A and the average value is zero.

The echo voltage produced at time t for N scatterers located in the range slice producing reverberation can be written as a sum of terms like those above -- so long as each scatterer is weak and multiple scattering effects can be neglected -- giving

$$V(t) = M\, P_o \sum_{l=1}^{N} R_l B(\theta_l,\phi_l)\ \exp -j(\omega t + kr_l)/r_l^2$$

or

$$V(t) \approx \frac{MP_o \exp(-jkr)}{r^2} \sum_{l=1}^{N} R_l B_l \exp(jF_l)$$

where in the second expression we have assumed r >> cτ/2 and simplified the notation for B. The phase term is rewritten as fluctuations around the mean value, kr, and denoted F. Under realistic measurement conditions it is reasonable to expect that the number of scatterers in a predefined range interval as well as their relative locations will vary over time. If the number of scatterers changes, the distributions of reflectivities will likely change as well. Thus, all of the quantities associated with the summation are variables; We assert, from experience, that they will be random variables and assume that, over measurement scales of interest, they are stationary as well. In general, linear transformation of stationary random variables yields a stationary random process, so we further assume that V(t) is a member of a set of stationary random variables.

Each measurement of the scattered pressure at time t from a representative arrangement and number of scatterers is unique, since the voltage is a random variable. Hence, it is appropriate to examine ensemble averages of replicated measurements of the reverberation. Using the form for a random sum of random variables we can calculate the expected value of the instantaneous scattered pressure as

$$\langle V(t)\rangle = M\, P_o \langle N\rangle \langle R\rangle \langle B\rangle \langle \exp(jF)\rangle / r^2$$

where we make the plausible assumption that the number, location, and reflectivities of the scatterers are uncorrelated. The notation <•> is used to denote expected value over an ensemble, thus <R> is the expected value of the reflectivity of all scatterers which might be found in the scattering region. The term <B> is the expected value of the effect of the beam pattern(s) of the transducer(s) due to the spatial arrangement of the scatterers. This term involves major assumptions to calculate explicitly. If the relative locations of the scatterers are uncorrelated (there is no rigid pattern), then the phases of the individual echoes will also be uncorrelated. This will invariably be the case in real measurements at sea and leads to the conclusion that the expected value of the instantaneous echo voltage is zero.

The instantaneous intensity of the reverberation is proportional to the square of the voltage, viz.

$$I(t) \quad V(t)\, V^*(t)$$

$$I(t) = (M/r^2)^2\, I_o \sum_{l=1}^{N} \sum_{m=1}^{N} R_l R_m\, B_l\, B_m\, \exp j(F_l - F_m)$$

where I_o is the intensity of the projected pulse. This expression can be expanded into terms for l=m and l≠m:

$$I(t) = I_o\, (M/r^2)^2 \left(\sum_{l=m}^{N} R_l^2 B_l^2 + \sum_{l \neq m}^{N} \sum R_l R_m B_l B_m \exp j(F_l - F_m) \right)$$

Taking the ensemble average, the expected value of the instantaneous reverberation intensity is

$$\langle I(t)\rangle = I_o (M/r^2)^2 \langle N\rangle \langle R^2\rangle \langle B^2\rangle + \langle N^2 - N\rangle \langle R\rangle^2 \langle B\rangle^2 \langle \exp j(F_l - F_m)\rangle$$

If we make the assumption again that the relative phases of the individual echoes are uncorrelated, then

$$<I(t)> \approx I_o(M/r^2)^2 <N> <R^2> <B^2>$$

and the average (RMS) scattered intensity is proportional to the average number of scatterers and their average scattering strength. If there is any degree of correlation between the scatterer locations, the second term may not be negligible and the simple result above will not hold. Experiments on schooled and/or caged fish suggest that this effect, if present, is overshadowed by attenuation effects at the edges of the school and/or multiple-scattering.

The second moment of the intensity can be calculated in the same manner as above. Skipping the details,

$$<I^2(t)> = I_o^2 (M/r^2)^4 \left(<N><R^4><B^4> + <2N^2 - N><R^2>^2<B^2>^2 \right)$$

if the relative phases are assumed uncorrelated. With the second moment of the intensity we can calculate the expected value of the variance,

$$\sigma_I^2(t) = <I^2(t)> - <I(t)>^2$$

$$= I_o^2 (M/r^2)^4 \{<N> <R^4> <B^4> + <N^2-N> <R^2>^2 <B^2>^2\}$$

The expressions for average value and variance of the scattered intensity involve various moments of N, the number of scatterers in the insonified volume, and B, the beampattern function evaluated at the locations of the (randomly distributed) scatterers. The moments of both quantities are linked through the spatial distribution function which describes the spatial locations of the scatterers. Explicit expressions for $<I(t)>$ and $\sigma_I^2(t)$ cannot be calculated until the moments of N and B can be determined. This requires assumption of a spatial distribution for the scatterers.

Note, however, that so long as the distribution of scatterers and scatterer reflectivities remains constant (over sequential measurements) **the mean intensity will be proportional to the number of scatterers**, although the constant of proportionality might change. The major effect of unusual distributions will be to increase the variance of the intensity measurements.

The most common hypothesis about spatial patterns of marine organisms is that they distribute within given volumes according to the Poisson distribution. Objects distributed in this way are said to be uniformly distributed in 3-space. The probability density function for the Poisson distribution is

$$P(N) = (\eta V) \exp(-\eta V)/N!$$

where P(N) is the probability of finding exactly N objects (scatterers) in a volume of space V and η is the average number of objects per unit volume. If we assume that the scatterers producing reverberation are distributed uniformly, at least within the regions of interest, then we can calculate the required moments of N and B. The moments of N are

$$<N> = \eta V$$

and

$$<N^2> = (\eta V)^2 + \eta V.$$

The moments of B can be calculated from

$$<B^n(\theta,\phi)> = \int_0^{\pi} \int_0^{2\pi} B^n(\theta,\phi)\ p(\theta,\phi)\ d\phi\, d\theta$$

where $p(\theta,\phi)$ is the probability density function of scatterers in volume elements, ΔV, centered at (θ,ϕ), normalized by the total number of scatterers in the spherical shell of volume

$$V = (4\pi/3)\ c\tau/2.$$

Since the number of objects in ΔV is $\eta\Delta V$ and the total number of objects is ηV, we can write

$$p(\theta,\phi)\ d\phi\, d\theta = \frac{\eta \Delta V}{\eta V} = \frac{\Delta V}{V}$$

or, in terms of the angular differentials

$$p(\theta,\phi)\ d\phi\, d\theta = \frac{3}{4\pi}\ \sin\theta\, d\phi\, d\theta.$$

Hence the moments of $B(\theta,\phi)$ are found from

$$<B^n(\theta,\phi)> = \frac{3}{4\pi} \int_0^{\pi} \int_0^{2\pi} B^n(\theta,\phi)\ \sin\theta\, d\phi\, d\theta.$$

Given a functional form for the directivity pattern of the combined response of a transducer, and assuming a uniform distribution of scatterers, we can calculate all the required moments of B.

Substituting for V and $<B^2>$ into the expression for the expected value of the scattered intensity, we obtain

$$<I(t)> = \frac{I_o M^2}{r^2}\ \eta <R^2> \frac{c\tau}{2} \int_0^{\pi} \int_0^{2\pi} B^2(\theta,\phi)\ \sin\theta\, d\phi\, d\theta.$$

Taking logarithms, this can be put into the sonar equation form

RL = SL + RS -20 log r + Sv + 10 log(cτ/2) + 10 log(4π) - Jv

where

RL = 10 log $V^2(t)$ is the reverberation level in dB,

SL = 10 log I_o is the source level,

RS = 10 log M^2 is the receiving sensitivity,

$$Sv = 10 \log (\eta \langle R^2 \rangle)$$ is the volume scattering strength,

and

$$Jv = 10 \log \left(\frac{4\pi}{\int_0^{\pi} \int_0^{2\pi} B^2(\theta,\varnothing) \sin\theta \, d\varnothing \, d\theta} \right)$$

is the directivity index for reverberation. This equation is identical to that presented by Urick (1967), a not surprising result since quite similar assumptions are made here. Note that the definition of Sv in terms of the average target strength and average numerical density of scatterers is valid in general, but calculation of Sv from measured scattering levels requires estimation of the beampattern effects. This correction factor is, in turn, dependent upon the spatial distribution assumed for the scatterers. Patchy and/or aggregated patterns of scatterers within resolved scattering volumes are inconsistent with this development and these results would not apply.

For practical transducers, there are simpler expressions for the last two terms of the equation for **RL**. For example, if the transducer is a circular piston element of diameter, D, then

$$\mathbf{RL} = SL + RS - 20 \log r + Sv + 10 \log(c\tau/2) + 10 \log(4.853/kD)$$

where k is the wavenumber, $k=2\pi f/c$. If the Directivity Index (DI) for the piston transducer is known, then we can write

$$\mathbf{RL} = SL + RS - 20 \log r + Sv + 10 \log(c\tau/2) + 7.7 - DI.$$

By heuristic arguments, we can derive the probability density functions for signals similar to volume reverberation echoes. These expectations would apply to the limiting case of volume scattering, with all of the assumptions above plus the additional assumption of relatively many scatterers in the insonified volume. These distributions can be considered as sufficient conditions for acceptance of an echo process as volume scattering but, as will be shown, they are not necessary. Testing against these distributions constitutes a conservative test for volume scattering.

Consider a general scattered field composed of a number M of independent contributions from individual scatterers. We can write the total field, E, as a sum of quadrature components

$$E = \sum_{i=1}^{M} x_i + j \sum_{i=1}^{M} y_i$$

and in polar form as

$$E = A \exp(jF).$$

By the central limit theorem, the probability density function of a sum of M independent random variables approaches the normal distribution as M grows large. If we assume that the quadrature components are independent random variables, whatever their distribution, then we can treat the sums of the quadrature components as random variables from a normal population. It is reasonable to further assume that these quantities are zero-mean and have equal variance. With the definitions

$$X = \sum_{i=1}^{M} x_i$$

$$Y = \sum_{i=1}^{M} y_i$$

$$A = \sqrt{X^2+Y^2}$$

$$F = \tan^{-1}(Y/X)$$

it is straightforward to show that the probability density function for the RMS amplitude, A, is the Rayleigh distribution

$$P(A) = \frac{A}{\alpha^2}\exp - (A^2/\alpha^2)$$

where α is a parameter, and that the phases are uniformly distributed over $-\pi/2$ to $\pi/2$. The mean and variance for A are easily calculated

$$\langle A\rangle = \alpha\sqrt{\frac{\pi}{2}}$$

$$\sigma_A = (2-\frac{\pi}{2})\alpha^2$$

The probability density function for the intensity, $I = A^2$, can be obtained from P(A)

$$P(I) = \frac{1}{2\alpha^2}\exp -(A^2/\alpha^2)$$

which is the exponential distribution. Some of the moments of the intensity are

$$\langle I\rangle = 2\,\alpha^2$$

$$\langle I^2\rangle = 8\,\alpha^4$$

and the variance is

$$\sigma_I = 4\alpha^4$$

It is occasionally useful to treat averages of log measures. The probability density function for S = 10 log (I) can be obtained (Mikhalevsky 1979) from the equations above, giving

$$P(S) = \frac{1}{2E\alpha^2}\exp\left(\frac{S}{E} - \frac{1}{2\alpha^2}\exp\left(\frac{S}{E}\right)\right)$$

where E =4.343.... is a constant. The mean and variance of S are

$$\langle S\rangle = \log_{10}(2\alpha^2) - E\,\gamma$$

$$\sigma_S = E^2\pi^2/6$$

where γ = .57721... is Euler's constant. It is interesting to note that the variance of log averages is a constant, independent of the Rayleigh parameter.

The degree to which actual volume scattering approaches the limiting case can be seen by inspecting comparable measures for both cases. For example, the ratio $\sigma_I/\langle I\rangle^2$ has the value 1 for the case of Rayleigh statistics. The value calculated from physical scattering is

$$\mathrm{VAR}\,(I)/\langle I\rangle^2 = 1 + \frac{1}{\lambda V}\left(1 + \frac{\langle R^4\rangle}{\langle R^2\rangle^2}\,\frac{\langle B^4\rangle}{\langle B^2\rangle^2}\right)$$

It is clear that this ratio will approach the limiting value of one as the number of scatterers increases, independently of the properties of the scatterers as expressed in the moments of R and B. In other words, **a uniform random distribution is not necessary to obtain Rayleigh statistics** for the scattered signals, although **we cannot calculate quantities like Sv without knowledge of the distribution.** In addition, this result also shows that the limiting value of variance is a minimum value. Experimental data would be expected to vary about this value only when the number of scatterers insonified is large.

Observation of a quantity like $\mathrm{VAR}\,(I)/\langle I\rangle^2$ is one way to experimentally verify that volume scattering conditions apply. So long as measured values of this quantity (or a related one) are within a statistically valid range about the expected value, scattering data can be assumed to arise from volume scattering.

The last equation also suggests a way to roughly estimate the number of scatterers, if plausible approximations for the distribution of reflectivities and spatial pattern can be made. For example, if the scatterers are assumed to be distributed according to the Poisson hypothesis, the factor $\langle B^4\rangle/\langle B^2\rangle^2$ can be calculated explicitly. Suppose the transducer were a circular piston with diameter/wavelength = D, then $\langle B^4\rangle/\langle B^2\rangle^2 = 2.54\ D^2$. The ratio $\langle R^4\rangle/\langle R^2\rangle$ might vary from 1 (constant reflectivity) to perhaps 6 (exponential distribution). Experiments on scattering from individual fish suggest that a Rayleigh distribution is not a bad fit to echo amplitudes, whence $\langle R^4\rangle/\langle R^2\rangle^2 \approx 2$ might be a reasonable estimate. Using these example estimates,

$$\lambda V = \frac{1 + 2 \bullet 2.54\ D^2}{\mathrm{VAR}\,(I)/\langle I\rangle^2 - 1}$$

This estimate can only be made when the variance ratio is greater than unity, obviously, corresponding to situations where the number of scatterers is not large. The attractive feature of this estimate is the independence from calibration factors of any kind, other than the assumptions about the nature and distribution of the scatterers. A similar exposition was published by Wilhelmij and Denbigh et al (1984) and Denbigh, et. al (1991).

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