Multiple-frequency Acoustical Estimation

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Abstract Remote estimation of abundances and size distributions of certain classes of marine organisms can, in principle, be made from acoustical scattering measurements at several frequencies. These estimates are obtained by solving an inverse problem involving the acoustical measurements and predicted (or measured) scattering coefficients for the organisms. The quality of acoustical size-abundance estimates depends upon several factors, including the accuracy of the measurements, the functional form and accuracy of the scattering model for the organism, the number and choice of measurement frequencies, and the solution algorithm. This paper describes the theoretical basis for multiple-frequency acoustical estimation and analyzes some of the problems involved in making accurate and precise estimates of size abundances.

Introduction

The use of multiple-frequency acoustical measurements to estimate size distributions of zooplankton was proposed several years ago by

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McNaught (1968) and, more recently, by Holliday (1977). The former used a specific scattering model for zooplankton to show that echosounders operating at different frequencies should be maximally sensitive to scattering from zooplankton of different sizes. He then suggested that differences in the echo levels at a pair of frequencies might be used to estimate biomass in a size range determined by the two frequencies. Holliday presented a formal mathematical method for estimation of size-abundance distributions from acoustical measurements at several frequencies given an accurate model for the scattering strength of individual targets.

Improved solution methods for the multiple-frequency scattering problem were obtained by Johnson (1977a) and used to estimate size distributions and total abundance of swimbladder fishes. These algorithms were later used by Greenlaw (1979) to estimate size-abundance profiles for euphausiids in a fjord and by Holliday (1976) to estimate distributions of swimbladder sizes for larval and juvenile anchovy. Agreement with independent samples of the scattering populations was quite good, demonstrating the power of this technique for at least some species of micronekton.

Application of multiple-frequency acoustical estimation to a specific target population is, conceptually at least, straightforward but significantly more complex than most bioacoustical applications. The (potential) precision and resolution of this method also requires more careful experimental design by the user. There are three basic"parameters" which the experimenter can vary to suit a particular situation: (*1*) the measurement frequencies, (*2*) the spatial coverage and resolution of the measurements (through the transducer directivities and method of deployment), and (*3*) the data processing algorithm. All of these must be chosen on the basis of some assumed characteristics of the target population (range of numerical densities, expected size distributions, spatial patterns, depth(s) of occurrence, etc.) and their usually complex relationships with acoustical measurements. Since this is still an experimental technique with some yet unresolved problems, we will attempt here only to describe the basic estimation problem, explore some relevant measurement factors that must be considered in designing a multiple-frequency acoustical system, and briefly discuss the mathematical estimation problem.

Basic Estimation Method

Zooplankton are, individually, very weak scatterers of sound. In most realistic situations, useful echoes from zooplankton will only be obtained when many individual zooplankters are simultaneously contributing to the total echo. When this is the case, it is usually assumed that the addition of the individual echoes is random in such a way that the total intensity of the scattering at some instant is, on average, equal to the sum of the intensities of the individual echoes contributing to the echo at that instant. Symbolically,

$$I_{\mathrm{T}}(t) = \sum_{l=1}^{N_{\mathrm{T}}(t)} I_l$$

where $I_{\mathrm{T}}(t)$ is the intensity of the scattering at time t and the I_l are the echo intensities of each of the $N_{\mathrm{T}}(t)$ scatterers insonified at time t. The echo intensities for the individual scatterers are functions of the frequency of the incident sound waves, the size (and perhaps other parameters) of the scatterers, their range, and the source level and directional properties of the transducer.

If the scatterers can be assumed to be uniformly and homogeneously distributed throughout the insonified volume, the effects of range and transducer directivity can be accounted for by applying a correction to $I_{\mathrm{T}}(t)$ (e.g. Urick, 1967 [p. 190–194]). The corrected intensity, $I'_{\mathrm{T}}(t)$, is then related to a sum [over a corrected number of scatterers, $N'_{\mathrm{T}}(t)$] of the actual echo intensities of the scatterers

$$I_{\mathrm{T}}' = \sum_{l=1}^{N_{\mathrm{T}}'(t)} I_l. \tag{1}$$

If the scatterers happened to have identical scattering strengths, I, we would have $I'_{\mathrm{T}}(t) = N'_{\mathrm{T}}(t)I$. It is more likely that the scattering strengths of the individuals will vary, however, particularly if there is a distribution of sizes present. A size distribution may be incorporated into the summation (Eq. 1), assuming the scattering strength dependence on size and frequency is known, in two ways.

If $N'_T(t)$ is large, we can assume a continuous size distribution of the form

$$N_c(a) \simeq N_T'(t)\ P(\mathrm{a})$$

where a is a characteristic size parameter. The function $P(a)$ is similar to a probability density function, having the property

$$\int_0^\infty P(a)\ \mathrm{d}a = 1.$$

We denote the scattered intensity at frequency f from scatterer of size a by $I_0(t)R\ (f,a)$ where I_o is the incident intensity. Then, since $N'_T(t)$ is large, the sum can be taken to a limit and we have

$$I_T'(t,f) \simeq I_0(t)\ \int_0^\infty R\ (f,a) N_c\ (a)\mathrm{d}a. \tag{2}$$

If $N_T'(t)$ is not large enough to justify the integral for Eq. 2 but sufficiently large that the random addition hypothesis is reasonable, or if the size distribution is discontinuous (e.g., organisms occur in discrete sizes), we can write the summation (Eq. 2) as

$$I_T'(t,f) \simeq I_0(t) \sum_{\mathrm{i}=1}^{s} N_d(a_i)R(f,a_i) \tag{3}$$

where s is the number of size classes.

Equation 2 is a Fredholm equation of the first kind. Methods exist to estimate the function $N_c(a)$ given measurements of $I_T'(t,f)$ at several frequencies and a knowledge of $R(f,a)$. These methods usually involve approximating the integral as a sum

$$I_T'(t,f) \simeq I_0(t) \sum_{i=1}^{s} W_i N_c(a_i)R(f,a_i)\Delta a \tag{4}$$

where W_i are weights derived from a quadrature formula. Since the discrete and continuous size distributions are related by

$$N_c\,(a_i)\,\Delta a \simeq N_d(a_i)$$

it can be seen that the integral form (Eq. 2) and discrete sum form (Eq. 3) differ only slightly. Thus the sum (Eq. 3) can be used as a prototype form.

The essence of the multifrequency estimation method is to apply equation 3 at a number of frequencies. For example, suppose we measure the relative scattered intensity ($\hat{I} = I_T'/I_0$) from a given volume at fixed range at F frequencies. Applying Eq. 3 for each frequency we have the set of equations

$$\begin{aligned}
\hat{I}\,(f_1) &\simeq N(a_1)R(f_1,\, a_1) + N(a_2)R(f_1,\, a_2) + \ldots + N(a_s)\, R(f_1,a_s)\\
\hat{I}\,(f_2) &\simeq N(a_1)R(f_2,\, a_1) + N(a_2)R(f_2,\, a_2) + \ldots + N(a_s)R(f_2\,,a_s)\\
&\;\;\vdots\\
\hat{I}\,(f_F) &\simeq N(a_1)\, R(f_F,a_1) + N(a_2)R(f_F,a_2) + \ldots + N(a_s)R(f_F,a_s). \quad (5)
\end{aligned}$$

These equations form a set of linear equations in the unknowns $N(a_i)$ with constant coefficients $R(f_j,a_i)$. Solution of this set yields an estimate of the size abundance of scatterers in the volume considered.

Unfortunately, solution of these equations is not a simple matter. From a strictly mathematical point of view, one must be concerned first with the existence of any solution at all [which is determined by the matrix of coefficients $R(f_j,a_i)$], next with the uniqueness of the solution (if $F > s$, there is no unique solution–although a "best" solution may be found), and then with the quality of the solution. These problems will be treated in a later section together with a general discussion of solution methods. The quality of the solutions, as will be shown later but is self-evident, is clearly a function of the quality of the input data [the measurements $\hat{I}(f_j)$ and the assumed scattering behavior of the scatterers contained in the coefficients $R(f_j,a_1)$].

There are three types of measurement error that can affect the solutions: random error, bias error, and what we might term validity error (which occurs when the conditions of measurement are such that model equations 5 are not valid). These errors will be described in the next section. Errors in the scattering strength coefficients certainly have an effect on the quality of the solutions also, but in a way that is difficult to quantify in general terms. Most scattering models for biological organisms are based upon idealized analyses or modest sets of measurements and may be insufficient for precise acoustical estimation using Eq. 5. Some comments on scattering models will be made following the discussion of measurement errors.

Acoustical Measurements

It is a fact of experience that sequential measurements of the scattered intensity from a region containing biological sound scatterers yield a sequence of fluctuating values. Theories to predict the statistics of this volume-reverberation process have been developed (e.g., Faure, 1964; Ol'shevskii, 1967; Middleton, 1967) and compared with measurements (Cron and Schumacher, 1961; Jobst and Smits, 1974). We can safely predict that the intensities we measure in order to apply the acoustical estimation techniques will be stochastic quantities, hence we anticipate the need to use averages of several measurements in Eq. 5. This in turn implies that our measured data will be subject to random error, due to finite sampling, and perhaps bias error as well.

The results of the theories can, with certain reservations, be used to predict the echo statistics for volume reverberation. If we make two important assumptions*, that the scatterers are uniformly randomly distributed throughout the insonified region, and that there are a large number of scatterers in the region, then it is straightforward to show that the echo amplitudes should follow a Rayleigh distribution, the intensities should follow an exponential distribution, and the log intensities should follow a modified exponential distribution (Dyer, 1970; Urick, 1977; Papoulis, 1965; Mikhalevsky, 1979). The probability density functions corresponding to these distributions allow calculation of mean values (where it is

*In the case of fairly strong scatterers such as fish, we must also assume that no multiple scattering occurs.

found that log intensities are biased estimates but amplitudes and intensities are not) and the population variances.

Given the population mean and variance of a random variable, x, we can estimate the 1-α level confidence interval for the expected mean from a set of N measurements by

$$CI(\overline{X}) = \mu_x[1 \pm \frac{t_{n-1}^{\alpha/2}}{\sqrt{N}} \; \frac{\sigma_x}{\mu_x}],$$

where t is Student's t parameter. We have calculated 95% confidence intervals for amplitude, intensity, and log-intensity averages as a function of N using this relationship and the appropriate population means and variances; the results are plotted in figure 1. The abcissa values are the total relative width of the confidence interval of the mean, $CI(\overline{X})/\mu_x$, in decibels. (It should be noted that confidence intervals expressed in decibels are not symmetric for amplitudes and intensities, although the differences are small for modest values of the confidence interval).

The figure shows two points of interest. First, it is apparent that averages of amplitudes, intensities, and log intensities produce about the same confidence interval, at least for averages of more than about 10 pings. Second, and most important, reduction of the uncertainty in the data estimate to very low levels will require a disproportionately large number of (independent) samples. For example, if we wish to attain ± 1 dB ($\backsimeq \pm$ 12%) uncertainty or less (for just this component of the random error) we must have over 70 samples of the scattering process for our averages. That is, at least 70 pings are required to achieve ± 1 dB uncertainty limits on the true scattered intensity.

This latter result has additional implications for sampling at sea. Implicit in the statistical theory is the assumption that the scatterer population is statistically stationary during the measurements. If the measurements are taken from a moving platform it is possible that changes in the parameters of the scattering population might occur during the time required to obtain the requisite samples for a desired precision. Should this occur, the sample variance will increase beyond the predicted value and, of course, the average intensity will no longer bear the assumed relationship to the number of

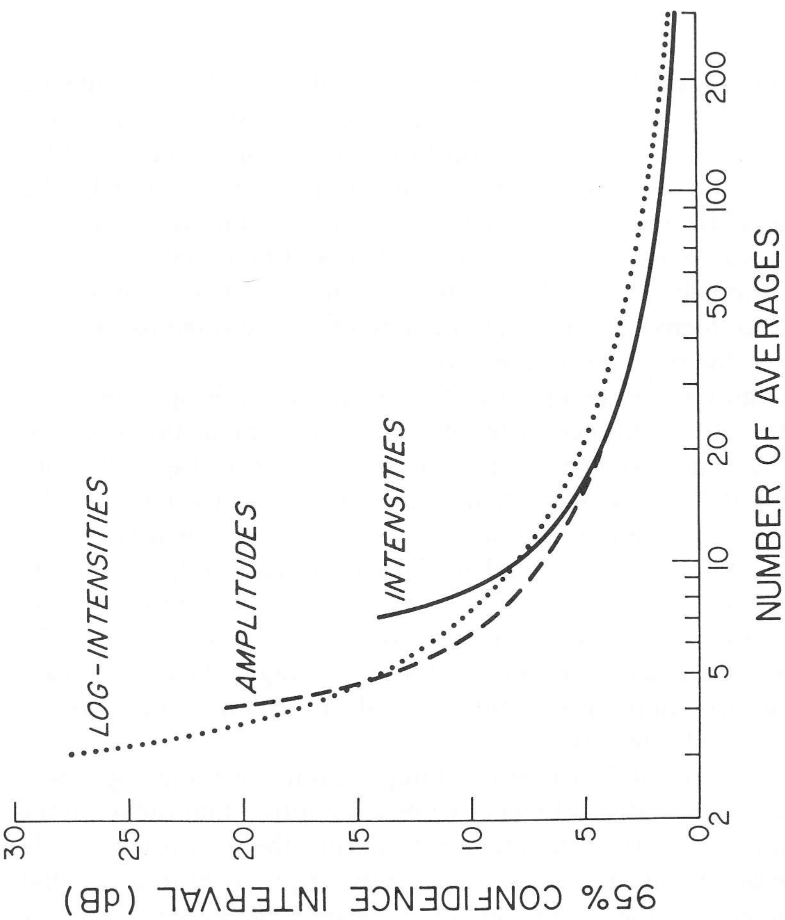

Figure 1. Widths of the 95% confidence intervals for finite averages of amplitudes, intensities (squared amplitudes), and log intensities of volume reverberation, expressed in decibels. Curves are drawn for aesthetic purposes only; the confidence intervals exist only at integral values of the abscissa.

scatterers. The horizontal scales of inhomogeneity for the scattering population will determine the magnitude of this effect. A partial solution is to increase the ping rate to the maximum practical extent or to reduce the ship speed. The potential for encountering inhomogeneous data is also increased by the requirement of measurements at several frequencies. Tests exist for determining the homogeneity and stationarity of data such as these (Middleton, 1969) and could, in principle, be applied at sea to create subsets of data suitable for further processing. However, this might impose an unwonted complexity on a practical acoustical system.

Another factor limiting both the accuracy and precision of acoustical measurements is noise. The reverberation echoes will be seen against a background of thermal (and, at low frequencies, wind wave and shipping) noise present in the water and noise will be added in the receiving electronics. This noise will generally add incoherently with the reverberation echoes, producing both bias errors and increased echo variance. If the total noise is Gaussian with rms intensity $\sigma^2{}_n$ and the reverberation echoes are Rayleigh with expected intensity I_R, then the expected value of the intensity of reverberation plus noise is

$$<I_{RN}> = I_R + \sigma_N^2 \tag{6}$$

if the reverberation and noise are uncorrelated.

With some calculation it can be shown that the variance of the total intensity is

$$\mathrm{Var}(I_{R+N}) = \mathrm{Var}(I_R)[1 + (4/S)+(2/S^2)] \tag{7}$$

where $S = [I_R/\sigma^2{}_N]$ is a measure of the signal-to-noise ratio (SNR). The factor in brackets is the increase in variance of the measured intensities over the expected value due to the presence of uncorrelated noise. This factor, expressed as percentage increase in Var(I), is plotted in Figure 2 versus the signal-to-noise ratio in decibels

$$\mathrm{SNR} = 10 \log [I_R/\sigma_N^2].$$

It can be seen that this increase in variance of the intensity is modest ($\leq \backsim$ 25%) so long as the SNR is at least 12 dB.

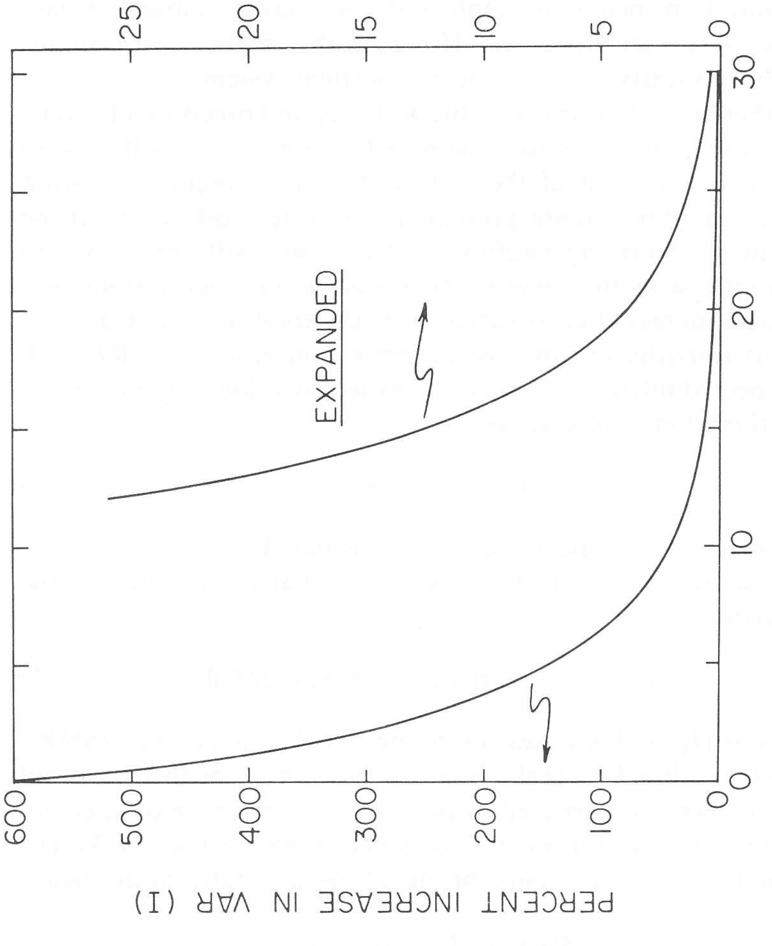

Figure 2. Increases in the variance of volume reverberation as a function of the (reverberation) signal-to-noise ratio in decibels. The noise is assumed Gaussian and uncorrelated with the reverberation.

The bias effect of noise is also low if the SNR is large. Even for modest SNR, the true value of the reverberation intensity may be estimated by subtracting an estimate of the noise intensity from the measured intensity. The validity of this estimate decreases with decreasing SNR, of course. However, this a posteriori accounting for noise effects is not useful for experimental design. It is known that ambient noise is frequency dependent and we are expecting frequency-dependent volume scattering strengths in order to produce our estimates of size abundance; thus we must anticipate the SNR to vary with frequency. SNR will also vary with the level of the scattering strength (i.e., with the number and size composition of the scatterers) and with the source level of the projecting transducer(s).

An example of the dependence of SNR on these factors is illustrated in Figure 3. The curves in this figure predict the value of SNR $-$ (SL $+$ 20 log τ), where SL is the projector source level and τ is the pulse length, for scattering from a population of euphausiids at 100-m range. We have used the size distribution of euphausiids measured at Saanich Inlet, B.C. (Greenlaw, 1979) and a scattering model for euphausiids (Johnson, 1977b; Greenlaw, 1977) to calculate volume scattering strengths for various densities (no./m^3) of euphausiids, ρ. Noise levels were taken from Urick (1967) for sea state 0. A circular piston transducer is assumed (the size is irrelevant) for these calculations, as is a matched bandwidth receiver. The horizontal dashed line indicates SNR = 0 for the combination SL = 105 dB//μbar, τ = 1 msec, and a receiver with a noise figure of 10 dB.

If a SNR of 10 dB is considered minimal for accurate measurements, it is clear from the figure that this combination of system parameters is inadequate at any frequency for euphausiid concentrations around $\rho = 1/\text{m}^3$. At densities of about 10/m^3 we would have adequate SNR over the frequency range 60–200 kHz. Higher densities yield wider frequency limits and higher SNRs, as one would expect. SNR may be increased by increasing the projector source level and by increasing the transmitted pulse length, but not indefinitely. Source level is limited by cavitation (Urick, 1967) to a practical maximum of about +125 dB//μbar for projectors located near the surface. The longest useful pulse length will be deter-

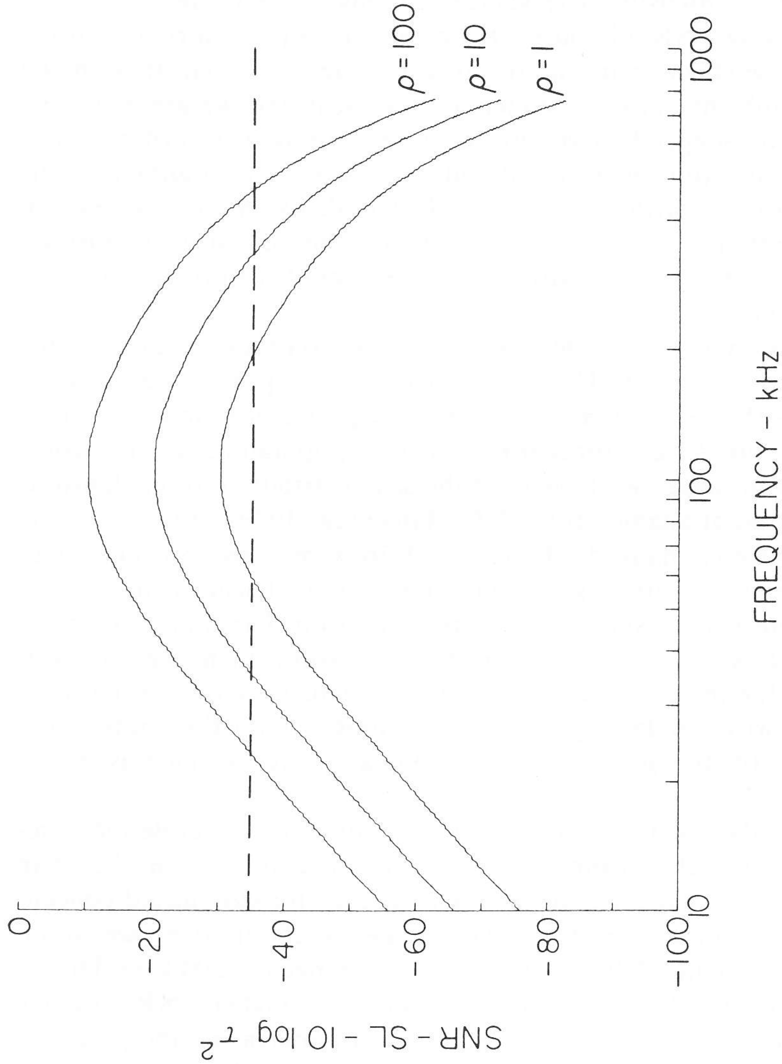

Figure 3. An example of the frequency dependence of the signal-to-noise ratio for a realistic case of scattering from populations of euphausiids (see text for details). The differences between the curves ($\rho = 1,10,100$ m^{-3}) and the dashed line are estimates of the signal-to-noise ratio for the system parameters noted in the text.

mined by the thickness of the scattering layers and the desired depth resolution. Increasing the pulse length beyond twice the layer thickness does not increase the SNR further.

The third type of error we noted involved the validity of the model equations (Eq. 5). It was explicitly noted that the basic assumption for the model was the presence of a large number of scatterers in the insonified volume, as this allowed representation of the total scattered intensity as the sum of the intensities scattered by the individual organisms, which in turn led to Eq. 5. If the number of scatterers insonified is not large, however, the intensity representation can no longer be made and the validity of the model equations is suspect. This is clearly the case for a single scatterer, where the echo statistics are dominated by the location of the scatterer in the beam rather than by the intensity-summation process. A reasonable question, then, is how many scatterers must be insonified to assure that Eq. 5 is valid?

This problem is entirely analogous to the statistical problem of when it is valid to apply the Central Limit theorem. Rules of thumb are often used to make a decision, with the magic number of independent random variables (i.e., scatterers) ranging from 5 to 30 or more. There is no strictly correct finite answer, of course. It can be shown that increasing the number of scatterers reduces the variance of the echo intensity closer to the theoretical value and the converse is also true. The approach to the theoretical value is asymptotic in much the same way as was shown for confidence intervals of average intensity versus the number of pings (Figure 1). The expression relating the excess intensity variance to the number of scatterers is complex, since it must include the effects of scatterers strengths (presumed random) and beam pattern effects, and nonlinear, so that sweeping generalizations are not possible. Sample calculations for a circular piston transducer at various directivities were made and it was found that the statistical rules of thumb were fairly adequate (increases in variance of no more than 25–50% for $N \geq 30$) if the insonified volumes were calculated from the -3-dB beamwidths of the transducer intensity pattern and the pulse length ($c\tau/2$). From this, one can estimate the minimum scatterer density (as a function of range) for which a given echosounder will yield useful echo statistics or select the directivity for an echosounder based upon expected minimum densities. This result also suggests that the maximum possible spatial resolution of volume scattering is dependent upon the scatterer density.

Scattering Models

The coefficients expressing the size-frequency dependence of the scattering from individual organisms, $R(f_j,a_i)$, are critically important to both the existence and quality of the solution estimate. No solution will exist if the matrix of scattering strength coefficients is singular and computed solutions will be unstable (subject to large variation as the input data changes slightly) if this matrix is nearly singular. The multifrequency estimation method will work poorly, if at all, for situations where the size and frequency dependence of the scattering strengths for an individual organism is jointly linear. For example, if $R(f,a) = g(f)a^2$ for all the measurement frequencies, it would not be possible to distinguish four scatterers of size a_0 from one scatterer of size $2a_0$. In this example the matrix of scattering strength coefficients would be singular. Best results should obtain when the population size and measurement frequency ranges are such as to yield a matrix of coefficients that are independent. The character of the size-frequency dependence of scattering strengths is an inherent property of the organisms, though. If a nonlinear region of size-frequency-dependent scattering strength exists for a particular organism then we may choose the measurement frequencies to exploit this. If no such region exists, these methods are not applicable.

Scattering strengths are not known precisely for any biological organisms, although some measured values are available for several disparate genera. Usually scattering strengths are estimated from a model, of which there are two distinct types: Empirical models, such as that developed by Love (1977) for fish, are essentially averages of many measurements expressed as functions of organism size, measurement frequency, and orientation of the organism. Conceptual models exploit some assumed physical similarities between organisms and certain geometric scatterers. A relevant example is the fluid-sphere scattering model (Anderson, 1950). This model has been applied with some success by Holliday and Pieper

(1980) to predict volume scattering strengths of copepods and, in modified form (Johnson, 1977b), to estimate size-abundances of euphausiids (Greenlaw, 1979).

Conceptual models are attractive for a number of reasons, including the ability to incorporate physical property variations in the model and the potential for describing many species with slight modifications of a prototype model. The continuous nature of a mathematical expression eases interpolation to frequencies or sizes not measured, with somewhat more confidence than a purely empirical model might permit. However, validating a conceptual model is not simple.

Neither type of model accounts for statistical behavior of the echoes. Empirical models explicitly use mean values, accounting for random variations of the scattering strength by assigning a confidence interval to the means. Conceptual models developed so far are wholly deterministic and observed variation in the echo levels merely serves to confound the comparisons of theory and experiment. The average measured volume-scattering intensity is proportional to the average (over size at a given frequency) scattering strength of the insonified organisms, thus mean values are necessary outputs from a scattering model. Where some of the experimental echo level variation is due to a behavioral characteristic of the organisms, however, such as aspect fluctuations for fish (Foote, 1980) or orientation changes for vertically migrating euphausiids (Sameoto, 1980; Greenlaw, 1977), the effective uncertainty as well as the mean value of the estimated scattering strengths can become both frequency dependent and time-of-day dependent.

The variance of the measured intensity contains the components of the variance of the scattering strengths (see Moose and Ehrenberg, 1971, for similar results for echo integration). In expressions for the relative variance, Var $(I)/\langle I\rangle^2$, the effect of scattering strength variance is inversely proportional to scatterer density. Thus we expect the random component of scattering strength to be important only when the density of scatterers is low, i.e., where we might be more concerned with the validity of our assumptions anyway.

Scattering model development constitutes the most serious technical impediment to application of multifrequency acoustical estimatiom at the present time. No model has been validated for any organism with sufficient precision to allow significant confidence in acoustically estimated abundances. The components of variation for scattering strengths are not well understood and are not generally incorporated in the models currently used. A good deal of work in the areas of model development and model validation is called for before the estimation of scattering coefficients can be considered accurate.

Solution Methods

The original integral relationship (Eq. 2) is a Fredholm equation of the first kind. It is well known (by Riemann's lemma) that this type of equation constitutes an ill-posed problem. Consider, for example, the equation $g(x) = \int_a^b K(x,y)f(y)\,dy$. To any function $f(y)$ we may add a perturbation $C\sin(wy)$ which, for suitable choices of w and for any C, produces very small changes in $g(x)$. Clearly the converse is true: very small errors in measuring $g(x)$ can potentially produce large variations in our estimates of $f(y)$. The degree to which these inverse solutions are unstable depends in part upon the functional form of the true $f(y)$, the kernel $K(x,y)$, and the solution method chosen. Generally, both the unknown function and the kernel are fixed (not under the experimenter's control to any significant extent) and thus one can only vary the solution method to minimize the inherent instabilities of the problem.

A number of somewhat different solution techniques have been developed for treating inverse problems similar to the one considered here. Certain of these are described (following generally the classification of Ishimaru, 1978) in the following sections.

Least Squares

We assume that the set of Eqs. 5 is a sampled version of the integral Eq. 2, which relates measured scattering intensities to the properties of the scatterers. Thus we can state the problem to be solved in the matrix form

$$\mathrm{RN} = X \tag{8}$$

where R is the $m \times n$ matrix of scattering-strength coefficients, N is the $n \times 1$ vector of unknown abundances, and X is the $m \times 1$ vector of measurements (scaled to absolute units). A straightforward solution for Eq. 8 can be written (Smith and Franklin, 1969; Jackson, 1972) as

$$N \simeq (R^{\mathrm{T}}R)^{-1}R^{\mathrm{T}}X \qquad \text{for } m \geq n$$

or (9)

$$N\ (R^{\mathrm{T}}(RR^{\mathrm{T}})^{-1}X \text{ for } m<n$$

where R^{T} is the transpose of R and $(\cdot)^{-1}$ denotes the inverse. For the case $m \geq n$ (more measurements than unknowns), Eq. 9 yields the classic least-squares solution which minimizes the residual

$$\varepsilon^2 = \varepsilon^{\mathrm{T}}\varepsilon = (RN-X)^{\mathrm{T}}(RN-X) \tag{10}$$

For $m<n$, Eq. 9 yields a solution that minimizes the euclidean length of the unknown vector, $||N||$. Note that a single solution is obtained in either case of Eq. 9.

For the case $m \geq n$ we can write an expression relating measurement errors, δX, to the errors in the estimate of N, δN (e.g. Ishimaru, 1978)

$$\left|\left|\frac{\delta N}{N}\right|\right| \leq ||\ (R^{\mathrm{T}}R)^{-1}\ || \cdot ||\ R^{\mathrm{T}}\ || \cdot \left|\left|\frac{\delta X}{X}\right|\right| \equiv \beta \left|\left|\frac{\delta X}{X}\right|\right|, \tag{11}$$

where

$$||N|| = \max |N_i| \qquad i = 1,2,\ldots,m$$

for the vectors N, δN, X, and δX and

$$||R^{\mathrm{T}}|| = \max_{\mathrm{i}} \sum_{j=1}^{n} |\ R^{\mathrm{T}}_{Yij}|$$

for the matrix factors. If the determinant $|R^{\mathrm{T}}R|$ is small, as would be the case if the elements of R are not very different, then the norm $||(R^{\mathrm{T}}R)^{-1}||$ can be quite large (orders of magnitude; see the example in Ishimaru, 1978) and errors in the measurements will be magnified in the solution estimate. Conversely, if the elements of R can be chosen (in our problem by choosing the measurement frequencies and the sizes at which we want to estimate abundances) in such a way as to maximize their differences, this magnification effect can be reduced. Note that $\beta \geq 1$, however; we cannot achieve estimates of greater precision than our data.

If the errors associated with the measurements X_i are zero-mean normal random variables with equal variance for each measurement frequency, then the residual (Eq. 10) is a measure of the fitting error of N to the true size abundance. The accuracy of our estimate is then dominated by the accuracy of the scattering model and the choices of frequencies and sizes, the precision by the measurement variance. We have shown previously that measurements of volume scattering contain a bias error (calibration errors and incoherent noise, Eq. 6) and that the measurement variances will not be constant (Eq. 7 and following discussion), however. Thus we must anticipate that the solution with minimum residual will not necessarily be the closest possible estimate to the actual size-abundance distribution.

Since the variance of a single measurement is indeterminant, it is clear that we must use the average of many measurements at each frequency as our input data. If we obtain similar average values for ambient noise (measured through the same system path as the reverberation echoes) then the bias effect of noise can be approximately removed by subtracting the rms noise intensity from the reverberation data (e.g., Eq. 6). In addition, if we estimate the measurement variance at each frequency it is feasible to approximate the equal variance situation by row scaling the vector X and the matrix R, viz. (Jackson, 1973)

$$\overline{X}_i = \frac{X_i - \eta_i}{S_i} \qquad i = 1,2,\ldots,m$$

where η_i is the noise intensity at frequency i and S_i is the measured standard deviation, and the scattering matrix elements

$$R_{ij} = \frac{R_{ij}}{S_i} \qquad i = 1,2,\ldots,m; \text{ all } j$$

The remaining components of bias error are those related to calibration. If these are negligible (in the context of Eq. 11, which is probably not the case), the least-squares solution to the scaled problem is the best-fit estimate the problem itself will allow.

In the situations likely to occur in practice (where calibration errors are $\backsim$ 5–10%, the scattering model is only approximate, and the mesh of frequencies and sizes are suboptimal) the inherent instabilities in the problem will probably produce inaccurate estimates. One dramatic way that inaccuracies may appear is the estimation of negative numbers of scatterers at some size or sizes. This sort of answer is physically impossible, of course, and merely signifies the presence of relatively large errors in the solution. When an experiment has been designed to minimize β (Eq. 11) and the data quality is apparently high, implausible results may serve as indicators that some aspect of the problem is incorrect. For example, scattering from fish can easily dominate that from zooplankton and solutions based upon a zooplankton-only scattering model usually will be strikingly implausible.

The possibility of estimating negative abundances can be eliminated by adding the following constraint to the problem:

$$N_i \geq 0, \text{ all } i. \tag{12}$$

This constitutes additional information about the unknown vector N and, in principle, should not reduce the accuracy of the solution estimate. Non-negativity is a fairly straightforward constraint for a least-squares problem. One algorithm for solving Eq. 8 with constraint Eq. 12 is NNLS (Lawson and Hanson, 1974). We have used this algorithm with some success to estimate size abundances of swimbladders in mesopelagic fishes (R. K. Johnson, unpublished) and of euphausiids (Greenlaw, 1979). NNLS is useful for under- as well as over-determined problems.

When there is reason to believe that the solution with minimum residual may not be the best solution, it is possible to modify a least-squares problem to allow more than a single solution. These solutions can then be inspected according to some criterion and the"best" of these selected. An example is the Levenberg-Marquardt analysis method where the original problem, Eq. 8, is modified by adding rows (Lawson and Hanson, 1974):

$$\begin{bmatrix} R \\ R_0 \end{bmatrix} N = \begin{bmatrix} X \\ X_0 \end{bmatrix} . \tag{13}$$

Generally we take $R_0 = \lambda I$, where λ is a positive parameter and I is the identity matrix. The addition to the data vector, X_0, can be taken as zeros, in which case we are expressing a preference that $||N||$ be small. Solutions to Eq. 13 depend upon the parameter λ. Generally, small values of λ lead to small values of ε^2 and large values of $||N||$ and β. Large values of λ tend to yield smaller values of $||N||$ and β, but larger residuals. Clearly this sort of analysis contains a measure of subjectivity in choosing among solutions. Since the practicality of choosing an optimum λ for each set of new data rests upon devising an algorithm relating λ, β, ε^2, and $||N||$, it is necessary to choose a selection criterion a priori. This subjective choice can often be given the guise, if not the substance, of objectivity (for example, by choosing λ such that ε^2 and $||N||^2$ are jointly minimum).

Regularization Methods

Regularization methods are similar to the least-squares method just described. The initial development by Phillips (1962) has been extended by Twomey (1963) and Tihonov (1963). The basic difference between regularization and least squares is the assumption that $N(a)$ is a smooth function. Regularization also explicitly provides for the input of a trial distribution, $N_0(a)$, which constitutes a guess about the true solution.

Starting with the problem, Eq. 8, regularization seeks a solution N which minimizes the quantity.

$$S^2 = (RN-X)^T(RN-X) + \gamma_1 (N-N_0) T(N-N_0) + \gamma_2 (BN)^T(BN) \tag{14}$$

where γ_1, γ_2 are non-negative parameters and B is a smoothing matrix. The first term of Eq. 14 is the least-squares residual, ε^2. The second term measures the difference between the solution, N, and the trial function, N_0, weighted by the parameter γ_1. The last term measures the smoothness of N ($BN = 0$ implies perfect smoothness) weighted by γ_2. Commonly, γ_1 is set to zero because a trial function cannot be obtained with any confidence. The solution in this case can be written ($m \geq n$)

$$N \simeq (R^T R + \gamma_2 B)^{-1} R^T X \tag{15}$$

and is a function of a single parameter (Chow and Tien, 1976).

As $\gamma_2 \rightarrow 0$, this solution approaches the least-squares solution of Eq.9. This is the case of no smoothing. Increasing γ_2 increases the smoothing effect of the matrix B. No objective method is known for choosing an optimum value of γ_2, which is a distinct drawback for use of this technique in acoustical estimation. The effect of γ_2 was investigated by Chow and Tien (1976) for an analytical situation. Interestingly, the values of γ_2 presented for various amounts of random noise added to the"measurements" ranged from 10 (no errors) to 5×10^7 (5% errors).

Another method that imposes smoothing on the solution is stochastic extension, developed by Franklin (1970). This technique explicitly recognizes the stochastic nature of both the measurements and the unknown abundance vector and bases a solution upon the (usually assumed) statistical properties of each. For example, if we can assume that the measurement errors and the abundance variability are independent, the inverse matrix for R comparable to Eqs. 9 and 15 is

$$(C_{NN} R^T)\,(R C_{NN} R^T + C_{XX})^{-1}$$

where C_{XX} is the covariance matrix of measurement errors and C_{NN} is the covariance matrix for the abundance vector. C_{XX} can often be measured but C_{NN} is generally unknown. Adroit choice of C_{NN} can reduce the error magnification factor, β (Eq. 11) considerably (Ishimaru, 1978), at least in principle. The major difference, in practice, between regularization and stochastic extension is the need to estimate a single parameter in the former and a matrix of parameters in the latter.

Backus-Gilbert Inversion

The inversion technique developed by Backus and Gilbert (1970) is a smoothing scheme, similar in some respects to regularization. A parameter, q, is used to obtain a family of solutions as in regularization. The unique feature of Backus-Gilbert inversion, however, lies in the fact that q is a bounded ($0 \leq q \leq 1$) parameter which directly controls the trade-off between resolution of the unknown function and the fitting error.

It is intuitively obvious that we cannot expect to obtain an exact representation of our unknown function, $N(a)$, from a finite set of measurements, $X(f)$. For a given number of independent measurements, information theory suggests that we should be able to accurately estimate about that number of independent points of $N(a)$. Attempting to estimate more points leads to a leakage phenomenon, where the estimate at a given point becomes a weighted average of the estimates at surrounding points. Similar smearing occurs when the data and/or size mesh are not independent.* In other inversion techniques the form of the weighting function that smears the resolution of the $N(a)$ is uncontrolled; it is implicitly set by the problem itself. Backus-Gilbert inversion differs by explicitly attempting to control this weighting function and, thus, the resolution of the solution.

Suppose we express the averaging or smearing process as

$$\overline{N}(a_j) = \int_{\alpha 0}^{\alpha 1} A(a,a_j)N(a)\mathrm{d}a \ , \ j=1,2,...,N \tag{16}$$

where $\overline{N}(a_j)$ is the weighted average of $N(a)$ at $a = a_j$ and $A(a,a_j)$ is some weighting function. As $A(a,a_j)$ approaches a delta function at a_j, the effect of neighboring values of $N(a)$ is decreased; correspondingly, the resolution of $N(a_j)$ is increased. We do not have complete freedom in choosing the weighting function, however. Backus and Gilbert (1970) assume that the $A(a,a_j)$ are a linear combination of the kernel function, $R(f,a)$

*This is determined by the scattering model and the choice of frequencies and sizes. For perfectly independent data we would have $\beta = 1$ in Eq. 11.

$$A(a,a_j) = \sum_{i=1}^{m} b_i \quad R(f_i,a_j) \qquad (17)$$

where the b_i are constants to be determined. Inserting Eq. 17 into Eq. 16 we obtain

$$\overline{N}(a_j) = \sum_{i=1}^{M} b_i \int_{\alpha 0}^{\alpha 1} R(f_i,a_j)N(a)da \qquad (18)$$

$$= \sum_{i=1}^{M} b_i X_i$$

which is a formal solution to the problem.

The choice of the coefficients b_i is assisted by defining two functions: a measure of resolution termed the spread function, $s(a_i)$, and the error variance $\sigma^2(a_i)$ (see Backus and Gilbert, 1970; Chow and Tien, 1976; Westwater and Cohen, 1973; and Ishimaru, 1978, for details), both of which are functions of the b_i. Then a weighted sum of the spread and error is formed

$$Q(a_j) = qs(a_j) + (1-q)w\sigma^2 (a_j)$$

where w is a constant chosen so that $s(a_j)$ and $w\sigma^2(a_j)$ are about the same magnitude over the a_j. The coefficients b_i are found by minimizing $Q(a_j)$ and solutions, $\overline{N}(a_j)$ obtained from Eq. 18. The parameter q allows a trade-off between errors (minimum variance) and resolution (minimum spread) as q is varied from 0 to 1.

Remarks

Successful application of multiple-frequency acoustical estimation has been shown to require (*1*) accurate and precise measurements of volume scattering from organisms; (*2*) the individual scattering behavior of these organisms must be accurately known; and (*3*) numerical inversion of these data must be accomplished by a stable and accurate inversion algorithm. We have attempted here to describe some of the more important factors that determine how well one can meet these requirements and suggest the critical areas for further development. Clearly, the major technical impediment to more widespread use of this technique is the lack of validated scattering models. This is largely a problem of obtaining sufficient measurements on important species. The difficulties involved are technical rather than conceptual and can be overcome. A less critical area is the development of satisfactory inversion algorithms which, it was noted, will require reasonably accurate scattering models. We forsee no major difficulties in this work either. Much work remains to be done to attain a reasonable degree of physical and mathematical rigor for this technique but, at this point, there appears to be no reason not to expect eventual success.

Once the mathematical consistency of multiple-frequency estimation is demonstrated (in the manner of Chow and Tien, 1976; or Westwater and Cohen, 1973, for example) it will still remain to demonstrate the accuracy of acoustical abundance estimates. This may prove to be unusually difficult. There is no practical way to determine absolutely the size-abundance distribution of a scattering population in realistic ocean environs (if there were, biologists would be using it and we would have little need for acoustical sampling) so, in one sense, the accuracy of any sampling method is indeterminate. Some form of accuracy demonstration will be required, however, before an acoustical technique will be acceptable to most biologists as a valid sampling tool. Such a demonstration might take the form of laboratory measurements of scattering from known populations of organisms, with comparisons of acoustically estimated and measured size abundances. It would be difficult to simulate all of the complexities of real-ocean assemblages in a laboratory tank, though, and it might in some ways be more convincing to compare acoustical estimates with some more conventional samples, such as those from nets.

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